Noncommutativity and the lightcone

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- With Tom Heinzl (Plymouth, UK),
  Mattias Marklund (Umeå, Sweden).
- J. Phys. A **40** (2007), 0704.3547[hep-th]
  and papers in 2009.

Outline

Two different limits:

1. Particle in a strong magnetic field.
2. Energy spectrum in (nearly) lightcone field theory.

Q. What do these limits have in common?
A. Unified by spectral flow.

Lightlike noncommutativity:

1. Particle in a strong crossed field.
2. Lightcone zero modes.

Particle in a magnetic field

- Non–relativistic particle, constant B–field in x^3 direction.

Landau, Z. Phys. 64 (1930)

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- Quantise: $[x^1, x^2] = \dfrac{i}{eB}$ and $H = V$.

Strong magnetic limit

- Co–ordinates of the (x^1, x^2) plane no longer commute.
- 'Peirels' substitution in $V(x^1, x^2)$ Peierls, Z.P. 80 ('33)

$$[x^1, x^2] = \frac{i}{eB} \to x^2 = \frac{-i}{eF}\frac{\partial}{\partial x^1} \, .$$

- Space–space noncommutativity. Snyder, P.R. 71 ('47)
- Same conclusion through Dirac brackets, etc.

Dunne and Jackiw NPPS 33 ('93), Guralnik et al. PLB 517 ('01), Jackiw NPPS 108 ('02)

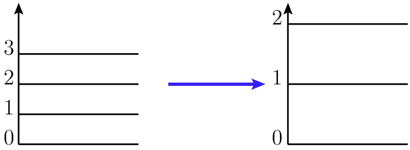
- To understand the limit, look at energy spectrum.

Landau levels

- At $V = 0$, Hamiltonian is a harmonic oscillator:

$$E(n) = \frac{eB}{m}\left(n + \frac{1}{2}\right) + \frac{p_3^2}{2m}\,.$$

- Landau levels infinitely degenerate w.r.t. p_1.
- In our limit: spectral gap between energy levels increases.



- Quantum hall effect:
higher occupation numbers in lower Landau levels.

Decoupling of states

1. As $B \uparrow$ (or $m \downarrow$), excited states become inaccessible.

   Decouple from the theory during spectral flow.

2. As $B \to \infty$ only lowest Landau level $n = 0$ level is available.

   Particle is confined to the states in $n = 0$.

3. Described by a planar noncommutative theory $[x^1, x^2] \neq 0$

   'Noncommutativity from spectral flow'.

T.H., A.I., JPA 40 ('07)

Toward the lightcone

- We now turn to our second example. We will see the same mechanism at work.

$$\mathcal{L}_2 = \partial_0\phi\partial_0\phi - (\nabla\phi)^2 \quad \longrightarrow \quad \mathcal{L}_1 = 4\partial_-\phi\partial_+\phi - (\nabla_\perp\phi)^2.$$

- Again, going from $2^{\text{nd}} \to 1^{\text{st}}$ order in time derivatives.

- We will work in 'almost light cone' co–ordinates. Chen, PRD 3 ('71)

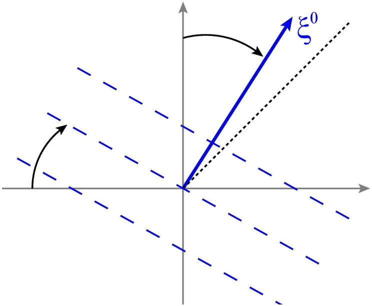
  Prokhvatilov and Franke, SJNP 49 ('89), Lenz et al. AP 208 ('91)

  Hellerman and Polchinski, PRD 59 ('99), Ilgenfritz et al. TMP 148 ('06)

- Investigate the limit of quantising on lightlike hypersurfaces.

Nearly lightcone co–ordinates

- Rotate into lightcone co–ordinates using parameter η.
- $\xi^0 = x^0(1 + \frac{\eta^2}{2}) + x^3(1 - \frac{\eta^2}{2})$.



- Time direction is ξ^0.
- Other directions x^-, $x^\perp$.
- $\{\xi^1, \xi^2\} = \{x^1, x^2\}$
- $\xi^3 = x^0 - x^3 \equiv x^-$.

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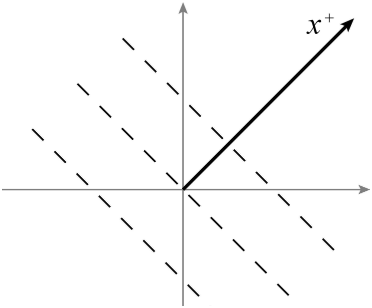

- As $\eta \to 0$, we recover lightcone:
- $\xi^0 \to x^+$.
- $\{\xi^1, \xi^2\} = \{x^1, x^2\}$
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Energy spectrum

- Look at spectrum of particle in our co–ordinates.
- On–shell energy k_0 from mass–shell, $k_\mu g^{\mu\nu} k_\nu = m^2$.

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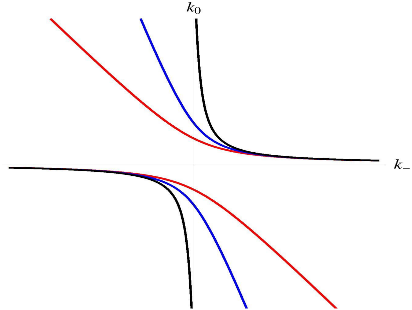
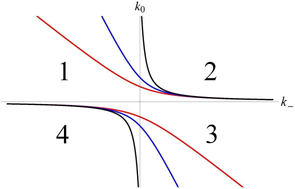


$$k_0 = -\frac{k_-}{\eta^2} \pm \sqrt{\frac{k_-^2}{\eta^4} + \frac{k_\perp^2 + m^2}{2\eta^2}}$$

- η decreasing: $\eta > \eta > \eta$.

- We should recover lightcone energies as $\eta \to 0$.

Spectral flow toward the lightcone



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- Label quadrants 1, 2, 3, 4.

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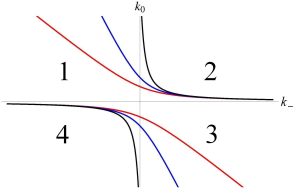

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$$k_0 = \frac{k_\perp^2 + m^2}{4k_-} + \mathcal{O}(\eta^2) \longrightarrow k_+ .$$

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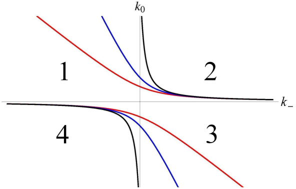

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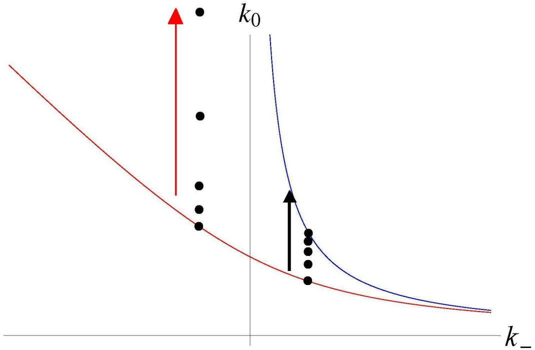
$$k_0 = \frac{k_\perp^2 + m^2}{4k_-} + \mathcal{O}(\eta^2) \longrightarrow k_+.$$

- In quadrants 1 and 3, energies blow up

$$k_0 = -\frac{2k_-}{\eta^2} - \frac{k_\perp^2 + m^2}{4k_-} + \mathcal{O}(\eta^2) \longrightarrow \infty.$$

Spectral flow toward the lightcone

- Half of the states become infinitely excited and decouple.
- Other half remain at finite energy – become lightcone states.



Noncommutative consequences

$$\mathcal{L} = 2\eta^2 \partial_0\phi\partial_0\phi + 4\partial_-\phi\partial_0\phi - (\partial_\perp\phi)^2 \,, \qquad \partial_0 \equiv \frac{\partial}{\partial\xi^0} \,.$$

- Quantising at $\eta \neq 0$ (with $\boldsymbol{x} = \{x^\perp, x^-\}$):

$$[\phi(\xi^0,\boldsymbol{x}), \pi(\xi^0,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y}) \,, \qquad [\phi(\xi^0,\boldsymbol{x}), \phi(\xi^0,\boldsymbol{y})] = 0 \,.$$

- Commutators $\leftarrow$ cancellations between Fourier modes.
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- Commutators $\leftarrow$ cancellations between Fourier modes.
- We know that half of the modes decouple!
- Loss of modes alters commutators – incomplete cancellation.
- At $\eta = 0$ field–field commutator no longer vanishes.

Noncommutative consequences

- Obtain lightcone field–field commutator:

$$[\phi(\xi^0, \boldsymbol{x}), \phi(\xi^0, \boldsymbol{y})] \longrightarrow$$

$$[\phi(x^+, \boldsymbol{x}), \phi(x^+, \boldsymbol{y})] = -\frac{i}{4}\delta^2(x^\perp - y^\perp)\text{Sign}(x^- - y^-)$$

- Different physics, but same mechanism as for magnetic field:

1. As $\eta \rightarrow 0$, high energy states decouple in a spectral flow.
2. Induces noncommutativity in configuration space.

The non–relativistic limit

- Non–relativistic limit of field theory. Zee, QFT... 2003
- States with $E > m$ become inaccessible.
- Klein–Gordon $\rightarrow$ Schrödinger equation:

$$\int d^4x\ \phi^\dagger(-\partial^2 - m^2)\phi \rightarrow \int d^4x\ \Phi^\dagger \left[i\partial_t + \frac{1}{2m}\nabla^2 \right] \Phi$$

- Noncommutativity in config. space: $[\Phi, \Phi^\dagger] \neq 0$.
- Including interactions:

✓ Only particle–number conserving interactions survive limit.

A.I., T.H., JPA 40 ('07)

Types of noncommutativity

- Space–space noncommutativity from magnetic fields.
- Other types of spacetime noncommutativity?

- 'Crossed fields': $|\mathbf{E}| = |\mathbf{B}| = F$ and $\mathbf{E}.\mathbf{B} = 0$.

  Low frequency approximation of laser fields. Vulcan, ELI, HiPER

$$\mathcal{S} = \int \mathrm{d}\tau \;\; -m\sqrt{\dot{x}^2} + eFx^+\,\dot{x}^1 \;, \qquad \dot{x} \equiv \frac{\mathrm{d}x}{\mathrm{d}\tau} \;.$$

- Limit $eF \gg m^2$ suggests "lightlike" noncommutativity:

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- Exception to the unitarity/ stringy problems of time–space noncommutativity. Aharony et al. JHEP 009 ('00)

Hamiltonian

- Relativistic particle Hamiltonians via τ gauge fixing.
- We can fix $x^+ = \tau$, the worldline parameter.

$$H = p_+ = \frac{(p_1 + eFx^+)^2 + p_2^2 + m^2}{4p_-}$$

- Expected $[x^1, x^+] \neq 0$, but $x^+ = \tau$, a smooth parameter.
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- No noncommutativity.
- But there are modes not propagated by this H:

  Zero modes of p_-.

Zero modes

- Look at mode propagators in field theory. Heinzl, LC2003
- Taking x^+ as time,

$$\Delta_{p_-,p_\perp}(x^+,y^+) = \begin{cases} \dfrac{\theta(p_-\delta x^+)}{4|p_-|} \exp -i \displaystyle\int_{y^+}^{x^+} \mathrm{d}u^+ \; H(u^+,p_-,p_\perp) \\ \\ \dfrac{i\delta(x^+-y^+)}{p_2^2+(p_1+eFx^+)^2+m^2-i\epsilon}, \quad p_-=0 \end{cases}$$

- $p_- \neq 0$ modes propagated by particle Hamiltonian – no noncommutativity.
- Do the zero modes see noncommutativity? Work in progress

Conclusions

- Many limits can be unified by spectral flow.
    - Strong magnetic field $\rightarrow$ planar noncommutativity.
    - Lightcone field theory – spectral flow.
    - Non–relativistic limit of field theory.
    - Noncommutative field theory from string theory.
- Lightlike noncommutativity from a crossed background field.
- Spectral flow?
- Tied to zero modes.

T.H, A.I, M.M, to appear