Temperature and confined fermionic systems

J. E. F. T. Ribeiro

Centro de Física das Interacções Fundamentais (CFIF),Departamento de Física, Instituto Superior Técnico, Av. Rovisco Pais, P-1049-001 Lisboa, Portugal

Introduction

This set of results is due to the effort of many authors among whom I cite:

- A. Nefediev
- S. Cotanch, A. Szczepaniak
- L. Ya. Glozman
- Pedro Bicudo, Gonçalo Marques, Ricardo
- F. Llanes Estrada
- G. Krein
- Yu Kalashnikova
- A. Vairo and N. Brambilla
- V. Beveren, G. Rupp and many others

Vacuum Structure in Strong Magnetic fields

The Hamiltonian of a relativistic fermion in an external field A_μ has the following form in 2+1 dimensions:

$$H = \int d^2x \, \bar{\psi}(x) \left[-i\gamma^j D_j + m \right] \psi(x) \; ,$$

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- Choose the Landau gauge $A_\mu = -By \, \delta_{\mu 1}$, where $B > 0$ is the magnetic field strength
- This problem is exactly solvable and constitutes a demonstration model for more complicated situations i.e. 3+1 dimensions
- We will use Bogolioubov-Valatin pseudo-unitary transformation to obtain the known results

$$_B\langle 0|\psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x})|0\rangle_B = -\frac{|eB|}{2\pi},$$

$$E_n = \sqrt{m^2 + 2n|eB|}$$

with n numbering the Landau levels

An example of Valatin-Bogolioubov Transformations

Three steps to construct the wave-function of a particle in a magnetic field from the wave-function of a free particle. From,

$$\psi(\mathbf{x}) = \sum_{\mathbf{p}} \frac{\mathbf{1}}{\sqrt{\mathbf{L_x L_y}}} \left\{ \mathbf{u}(\mathbf{p})\, \mathbf{a_p} + \mathbf{v}(\mathbf{p})\, \mathbf{b}^{\dagger}_{-\mathbf{p}} \right\} \mathbf{e}^{\mathbf{i p \cdot x}}$$

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- $u(\mathbf{p}) = \sqrt{\frac{\mathbf{E_p+m}}{\mathbf{2E_p}}} \begin{bmatrix} 1 \\ \frac{p_y - ip_x}{E_{\mathbf{p}} + m} \end{bmatrix} ; \mathbf{v}(\mathbf{p}) = \sqrt{\frac{\mathbf{E_p+m}}{\mathbf{2E_p}}} \begin{bmatrix} -\frac{p_y + ip_x}{E_{\mathbf{p}} + m} \\ 1 \end{bmatrix}$
- $\left\{ a^{\dagger}_{\mathbf{p}}, a_{\mathbf{p}'} \right\} = \left\{ b^{\dagger}_{\mathbf{p}},\ b_{\mathbf{p}'} \right\} = \delta_{p_x p'_x} \delta_{p_y p'_y},\ \ E_{\mathbf{p}} = \sqrt{m^2 + |\mathbf{p}|^{\mathbf{2}}}.$

The u and v spinors are the solutions of the Dirac equation for positive and negative energy respectively. (with $\cos\phi = \sqrt{\frac{E_{\mathbf{p}}+m}{2E_{\mathbf{p}}}},\ \sin\phi = \sqrt{\frac{E_{\mathbf{p}}-m}{2E_{\mathbf{p}}}}$.)

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- Step 1: preform a canonical transformation given by,

$$\begin{bmatrix} \tilde{a}_\mathbf{p} \\ \tilde{b}^{\dagger}_{-\mathbf{p}} \end{bmatrix} = R_\phi(\mathbf{p}) \begin{bmatrix} a_\mathbf{p} \\ b^{\dagger}_{-\mathbf{p}} \end{bmatrix} \qquad \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \mathbf{R}^*_\phi(\mathbf{p}) \begin{bmatrix} u(\mathbf{p}) \\ v(\mathbf{p}) \end{bmatrix}$$

- $R_\phi(\mathbf{p}) = \begin{bmatrix} \cos\phi & -\sin\phi\,(\hat{p}_y + i\hat{p}_x) \\ \sin\phi\,(\hat{p}_y - i\hat{p}_x) & \cos\phi \end{bmatrix},\ \hat{\mathbf{p}} = \frac{\mathbf{p}}{|\mathbf{p}|}$

An example of Valatin-Bogolioubov Transformations II

- The vacuum associated to the new operators $\tilde{a}$ and $\tilde{b}$ is given by
$|\tilde{0}\rangle = S|0\rangle = \prod_{\boldsymbol{p}}(\cos\phi + \sin\phi\, a^{\dagger}_{\boldsymbol{p}} b^{\dagger}_{-\boldsymbol{p}})|0\rangle$
- $\tilde{a}_{\boldsymbol{p}}|\tilde{0}\rangle = 0 \quad , \quad \tilde{b}_{\boldsymbol{p}}|\tilde{0}\rangle = 0$

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- Think of $\psi(\mathbf{x}) = \sum_{\mathbf{p}} \frac{\mathbf{1}}{\sqrt{\mathbf{L_x L_y}}} \left\{ \mathbf{u}(\mathbf{p})\, \mathbf{a_p} + \mathbf{v}(\mathbf{p})\, \mathbf{b}^{\dagger}_{-\mathbf{p}} \right\} \mathbf{e}^{\mathbf{i p \cdot x}}$ as an inner product between the Hilbert space spanned by the spinors $\{u, v\}$ and the Fock space generated by $\{a, b\}$
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- The choice of ϕ is to ensure the new spinors $\tilde{u}$ and $\tilde{v}$ are momentum independent:
$\tilde{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so that all the momentum dependence of ψ is stored in
$\{\tilde{a}_{\mathbf{p}}, \tilde{b}_{\mathbf{p}}\} = S\{\hat{a}, \hat{b}\}S$

Landau Levels

- Get the Landau Level representation
- Use $e^{ip_y y} = e^{-i\ell^2 p_x p_y}\sqrt{2\pi}\sum_{n=0}^{\infty} i^n \omega_n(\xi)\,\omega_n(\ell p_y)$
  $\omega_n(x) = (2^n n!\sqrt{\pi})^{-1/2} e^{-x^2/2} H_n(x)$
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- The wave function in the new basis can be written in the following way:
  $$\psi(\boldsymbol{x}) = \sum_{n\,p_x} \frac{1}{\sqrt{\ell L_x}} \left\{ \hat{u}_{np_x}(y)\,\hat{a}_{np_x} + \hat{v}_{np_x}(y)\,\hat{b}^\dagger_{n-p_x} \right\} e^{ip_x x}$$
- $$\begin{bmatrix} \hat{a}_{np_x} \\ \hat{b}^\dagger_{n-p_x} \end{bmatrix} = \sum_{p_y} \frac{i^n\sqrt{2\pi\ell}}{\sqrt{L_y}} \begin{bmatrix} \omega_n(\ell p_y) & 0 \\ 0 & -\omega_{n-1}(\ell p_y) \end{bmatrix} \begin{bmatrix} \tilde{a}_{\boldsymbol{p}} \\ \tilde{b}^\dagger_{-\boldsymbol{p}} \end{bmatrix}$$
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- The new operators satisfy the anticommutation relations:
  $\left\{ a^{\dagger}_{np_x}, a_{n'p'_x} \right\} = \left\{ b^{\dagger}_{np_x}, b_{n'p'_x} \right\} = \delta_{nn'}\,\delta_{p_x p'_x}$
- The vacuum is invariant under this change of basis, i.e., $\hat{a}_{np_x}|\tilde{0}\rangle = 0 \quad , \quad \hat{b}_{np_x}|\tilde{0}\rangle = 0$

An example of Mass Gap Equation

There are several approaches one can adopt to obtain the mass gap equation:

- 1-It can be derived as the condition for the vacuum energy to be a minimum, or,
- 2-getting rid of anomalous Bogolioubov terms or,
- 3-in the form of a Dyson equation for the fermion propagator, or
- 4-as a Ward identity.

Here we use 2. We have with $\cos\theta_n = \sqrt{\frac{E_n+m}{2E_n}} \quad , \quad \sin\theta_n = \sqrt{\frac{E_n-m}{2E_n}}$

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- The θ_n angles are to be found by imposing the vanishing of the anomalous terms in the Hamiltonian. A simple algebraic computation yields the following mass gap equations,
$$\begin{cases} (\ell\, m \cos\theta_0 + \sin\theta_2/\sqrt{2}) \sin\theta_0 = 0\,, & n = 0\,, \\ \ell\, m \sin 2\theta_n - \sqrt{2n}\cos 2\theta_n = 0\,, & n > 0\,, \end{cases}$$
- For any n have the following solution: $\tan 2\theta_n = \frac{\sqrt{2n|eB|}}{m}$, $E_n = \sqrt{m^2 + 2n|eB|}$

Vacuum Condensates and 3+1

It remains to construct the vacuum state in a magnetic field $|0\rangle_B$, annihilated by the operators a_{np_x} and b_{np_x} :

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- Next we consider the problem of dynamical symmetry breaking in the presence of the magnetic field.
  We obtain ${}_B\langle 0|\psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x})|0\rangle_B = -\frac{|eB|}{2\pi}$
- A fermion condensate occurs even in the absence of any additional interaction between fermions. This is an inherent property of the 2+1 dimensional Dirac theory in an external magnetic field.

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- In 3+1 Dimensions with quartic interactions these very same 3-steps can be performed

A class of Hamiltonians

For this talk let us consider the simplest Hamiltonian containing the ladder-Dyson-Schwinger machinery for chiral symmetry.

In any case most of the results presented here do not depend on the kernel choice

$$H = \int d^3x\, q^+(x)\left(-i\overrightarrow{\alpha}.\overrightarrow{\nabla}\right) q(x) + \int \frac{d^3x\; d^3y}{2} J^a_\mu(x) K^{ab}_{\mu\nu}(x-y) J^b_\nu(y)$$

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With,

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This class of Hamiltonians has a rich structure enabling the study of a variety of hadronic phenomena controlled by global symmetries

- It is Chiral compliant: The fermions know about the kernel !
- Reproduces in a non-trivial manner the low energy properties of pion physics like, for instance, $\pi - \pi$ Weinberg results for the scattering lengths together with Oakes-Renner, Goldberger-Treiman....
- Possesses the mechanism of pole-doubling in what concerns scalar decays

Bogolioubov Transformations

We can rotate the creation and annihilation Fock space operators. It is canonical !

- $$|\widetilde{0}>= Exp\left\{\widehat{Q}_0^+ - \widehat{Q}_0\right\}|0>$$

- $$\widehat{Q}_0^+(\Phi) = \sum_{cf}\int d^3p\,\Phi(p)\,M_{ss'}(\theta,\phi)\widehat{b}^+_{fcs}(\overrightarrow{p})\,\widehat{d}^+_{fcs'}(-\overrightarrow{p})$$

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With, the $^3P_0$ Coupling (Parity +):

$$M_{ss'}(\theta,\phi) = -\sqrt{8\pi}\sum_{m_l m_s}\begin{bmatrix} 1 & 1 & |0 \\ m_l & m_s & |0 \end{bmatrix}\times\begin{bmatrix} 1/2 & 1/2 & |1 \\ s & s' & |m_s \end{bmatrix} y_{1m_l}(\theta,\phi)$$

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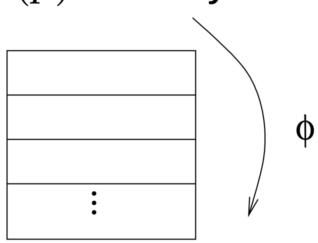
The functions $\Phi(p)$ classify the infinite set of possible Fock spaces:



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- Then we can consider the fermion field $\Psi_{fc}(\overrightarrow{x})$ as an inner product between the hilbert space spanned by the spinors {u,v} and the Fock space spanned by the operators {$\widehat{b}$,$\widehat{d}$}:

$$\Psi_{fc}(\overrightarrow{x}) = \int d^3p \left[u_s(p)\, b_{cfs}(\overrightarrow{p}) + u_s(p)\, d^{+}_{cfs}(\overrightarrow{-p})\right] e^{i\,\overrightarrow{p}\,.\,\overrightarrow{x}}$$

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- So that requiring invariance of $\Psi_{f\,c}(\overrightarrow{x})$under the Fock space rotations, is tantamount to require a counter-rotation of the spinors u and v,

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$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi M^{*}_{ss'} \\ \sin\phi M_{ss'} & \cos\phi \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

- The {u,v}, contain now the information on the angle $\phi(p)$.

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- ...this state should be degenerate with the vacuum. However in the chiral limit it turns out that such a pion-like excitation is non-normalizable !

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We have several ways of arriving at the mass gap equation

- $\widehat{H}_2[anomalous] = 0$
- Minimization of $H_0 : \frac{\delta H_0}{\delta\varphi} = 0.$
- Ward identities

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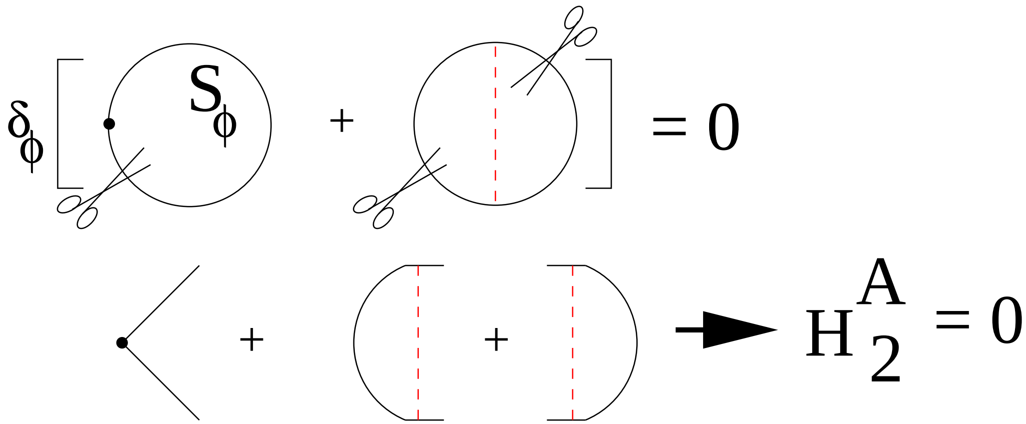

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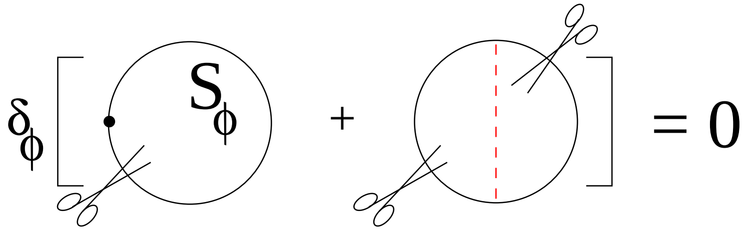

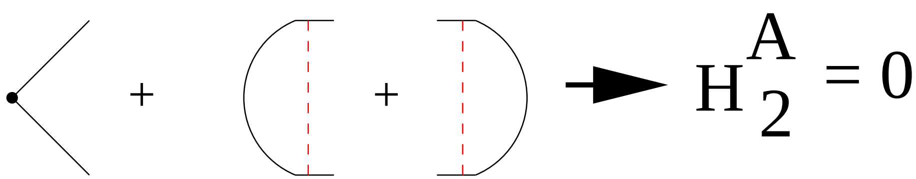

- $\Gamma_\mu(p,p^{'}) = \gamma_\mu + i\int \frac{d^4q}{(2\pi)^4} K(q)\Omega\, S(p^{'}+q)\Gamma_\mu(p^{'}+q,p+q)\Omega S(p^{'}+q)$
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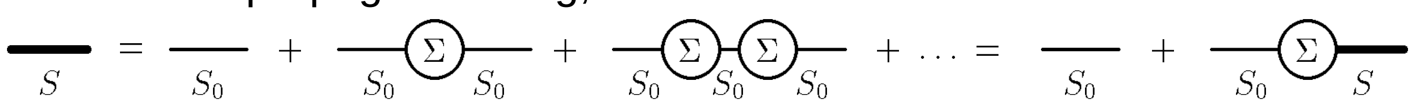

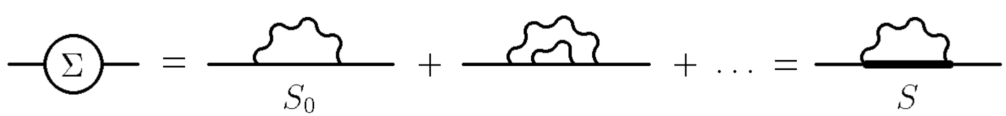

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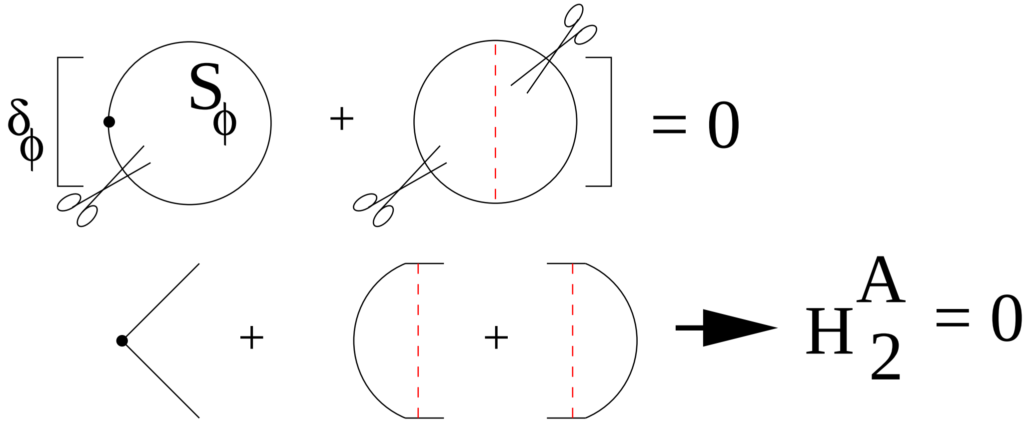

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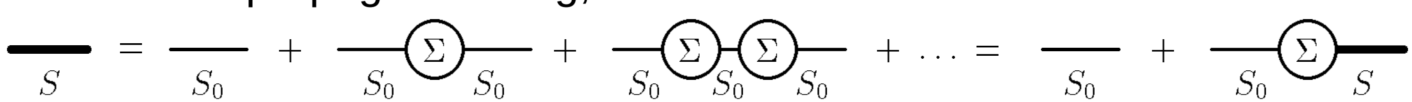

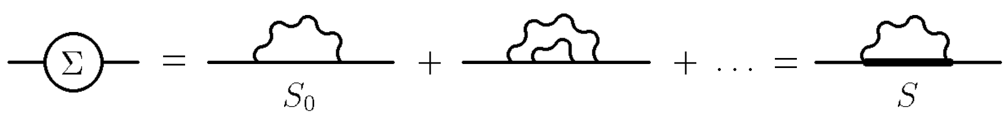

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- We have a nonlinear equation for the mass operator Σ,
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- the functions A_p and B_p represent the scalar part and the space–vectorial part of the effective Dirac operator.

- Finally $\tan\varphi_p = \frac{A_p}{B_p}$
$\varphi_{p\to\infty} \to 0$: only the vectorial part survives
$\varphi_{p\to 0} \to \pi/2$: only the scalar part survives

Breakdown of the expansion for φ_p in powers of $\hbar$

- The mass gap equation $A_p \cos\varphi_p - B_p \sin\varphi_p = 0$, with

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- As $\hbar$ vanishes, the chiral angle gets steeper and steeper at the origin, approaching the trivial solution $\varphi_p = 0$ for all p's and, in the limit of $\hbar \to 0$, we cease to have a low–momentum expansion of φ_p.

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- 1-Introduce dimensionless variables in the integral, $\vec{p} = \mu c\vec{x}$ and $\vec{k} = \mu c\vec{y}$; 2-define μ such that the resulting equation defines only the profile of the chiral angle and does not contain any scale at all. We have $\mu = \sqrt{\sigma\hbar c}/c^2$. Expand φ_p in low–momentum,
  $\varphi_p \underset{p\to 0}{\approx} \frac{\pi}{2} - \mathrm{const}\frac{pc}{\mu c^2} + \ldots = \frac{\pi}{2} - \mathrm{const}\frac{pc}{\sqrt{\sigma\hbar c}} + \ldots.$
- As $\hbar$ vanishes, the chiral angle gets steeper and steeper at the origin, approaching the trivial solution $\varphi_p = 0$ for all p's and, in the limit of $\hbar \to 0$, we cease to have a low–momentum expansion of φ_p.
- This was to be expected since we cannot build an action $\mathcal{S}$ out of the string tension σ and the speed of light c to obtain an expansion $\varphi_p = \frac{\hbar}{\mathcal{S}} \times f_1(p) + \frac{\hbar^2}{\mathcal{S}^2} \times f_2(p) + \ldots$

$\hbar$ expansions: $m \neq 0$

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- In conclusion we have two different regimes according to the parameter $m/\sqrt{\sigma}$: The spontaneous breaking of chiral symmetry is relevant for $m \ll \sqrt{\sigma}$, with heavy quark physics relevant for the opposite

Summary on V.B. transformations

The true vacuum, with the minimal vacuum energy, contains an infinite set of strongly correlated $^3P_0$ quark-antiquark pairs

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- The V.B. transformations constitute an Abelian Group:
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From the renormalized propagator we can construct the Bethe Salpeter equation for mesonic states. We can proceed via two identical formalisms

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In the space of pions we can write the Hamiltonian as,

$$H = \sigma_3 \begin{bmatrix} \Phi^{+} \\ \Phi^{-} \end{bmatrix} m_\pi \begin{bmatrix} \Phi^{+}, & \Phi^{-} \end{bmatrix} \sigma_3 + \sigma_3 \begin{bmatrix} \Phi^{-} \\ \Phi^{+} \end{bmatrix} m_\pi \begin{bmatrix} \Phi^{-}, & \Phi^{+} \end{bmatrix} \sigma_3$$

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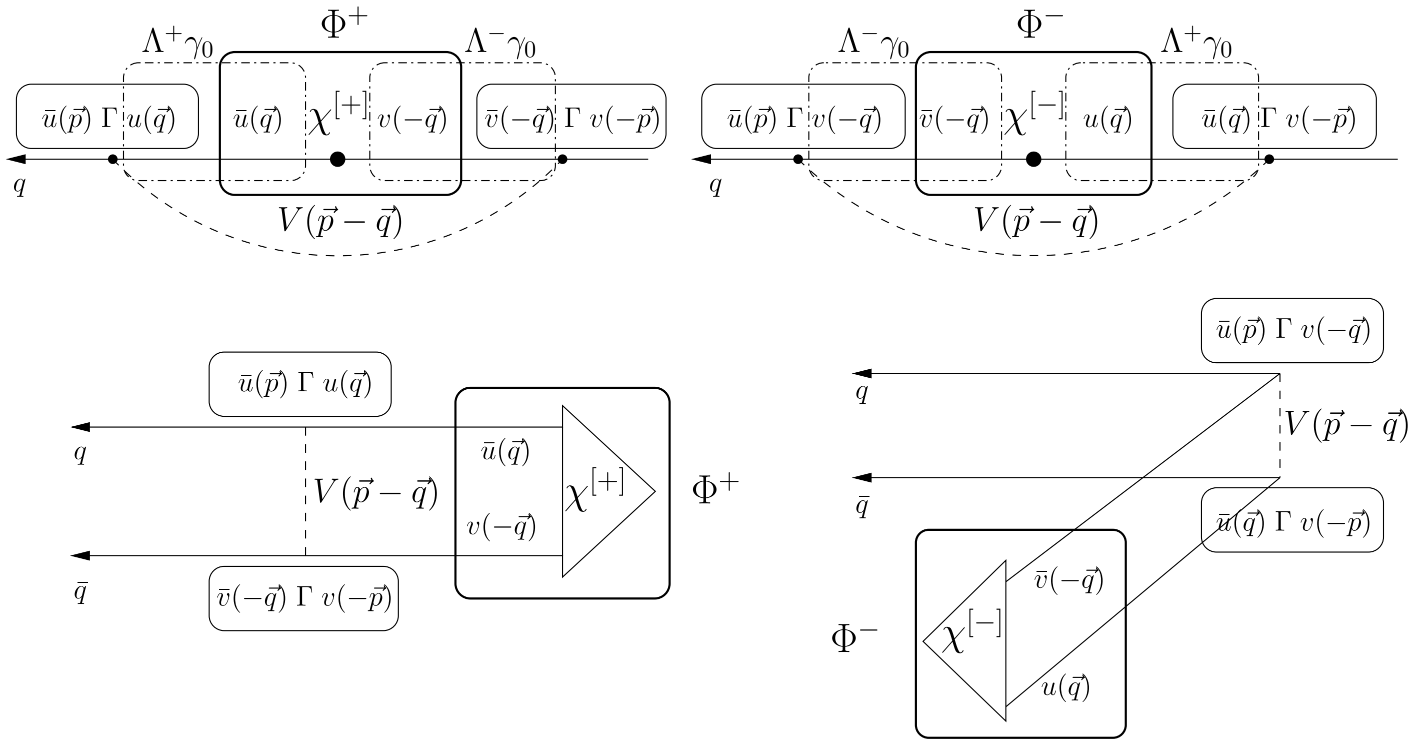

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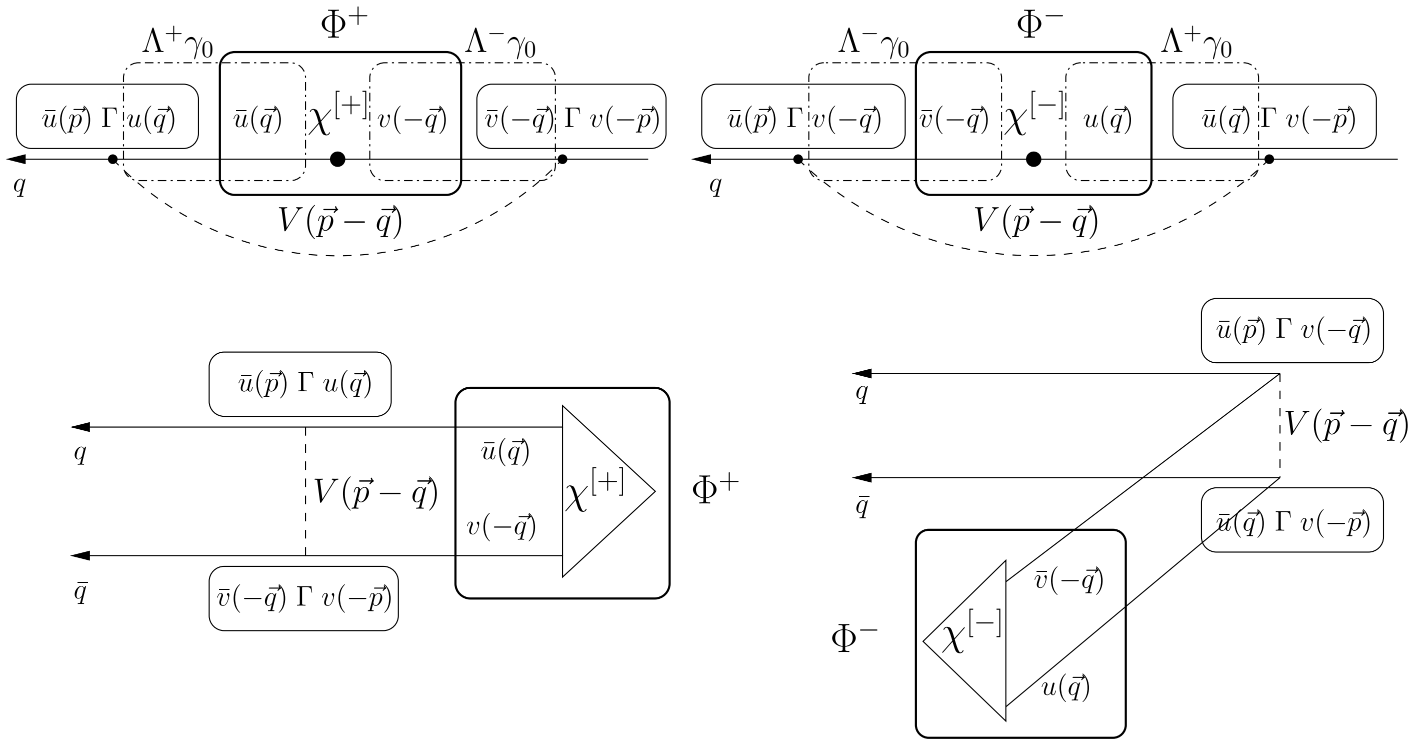

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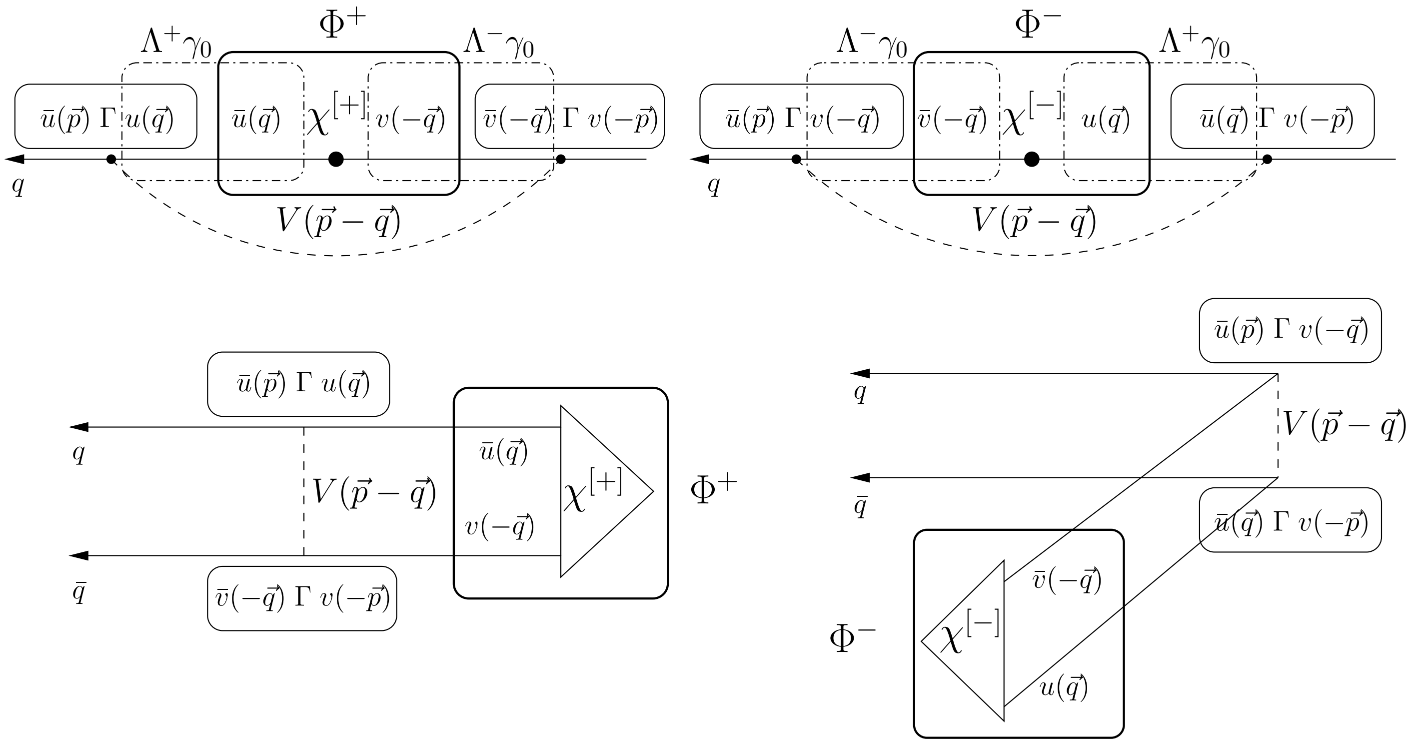

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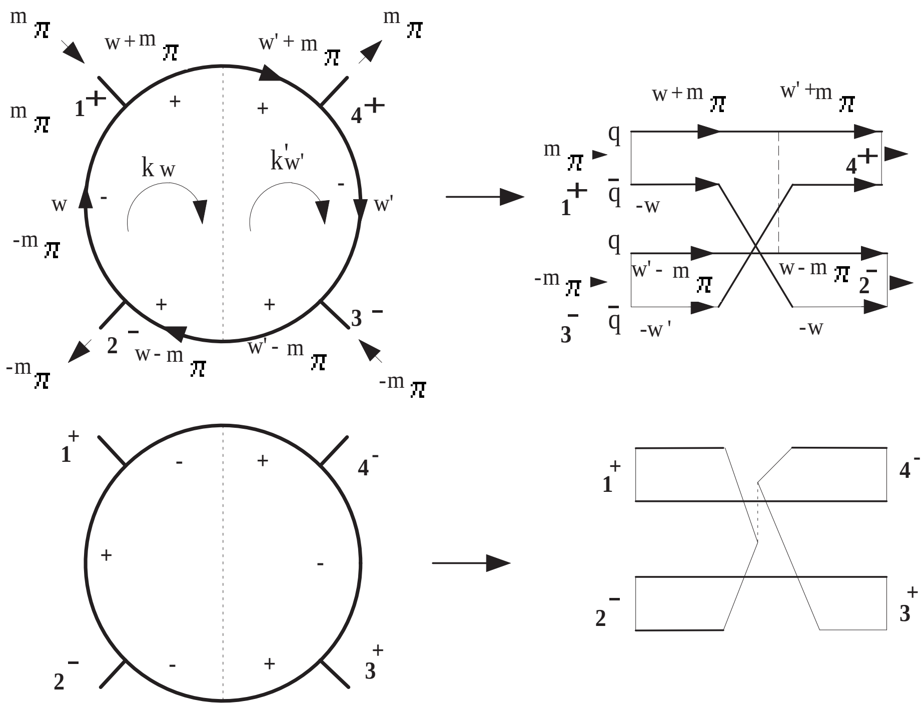
- We obtain $[\hat{m}_\pi,\ \hat{m}_\pi^\dagger] = \int \frac{p^2 dp}{(2\pi)^3}\left[\varphi_\pi^{+2}(p) - \varphi_\pi^{-2}(p)\right] = 1$,the normalization of the Pion Salpeter amplitudes

π-π scattering

The Weinberg formula for the scattering lengths of pions. We have,



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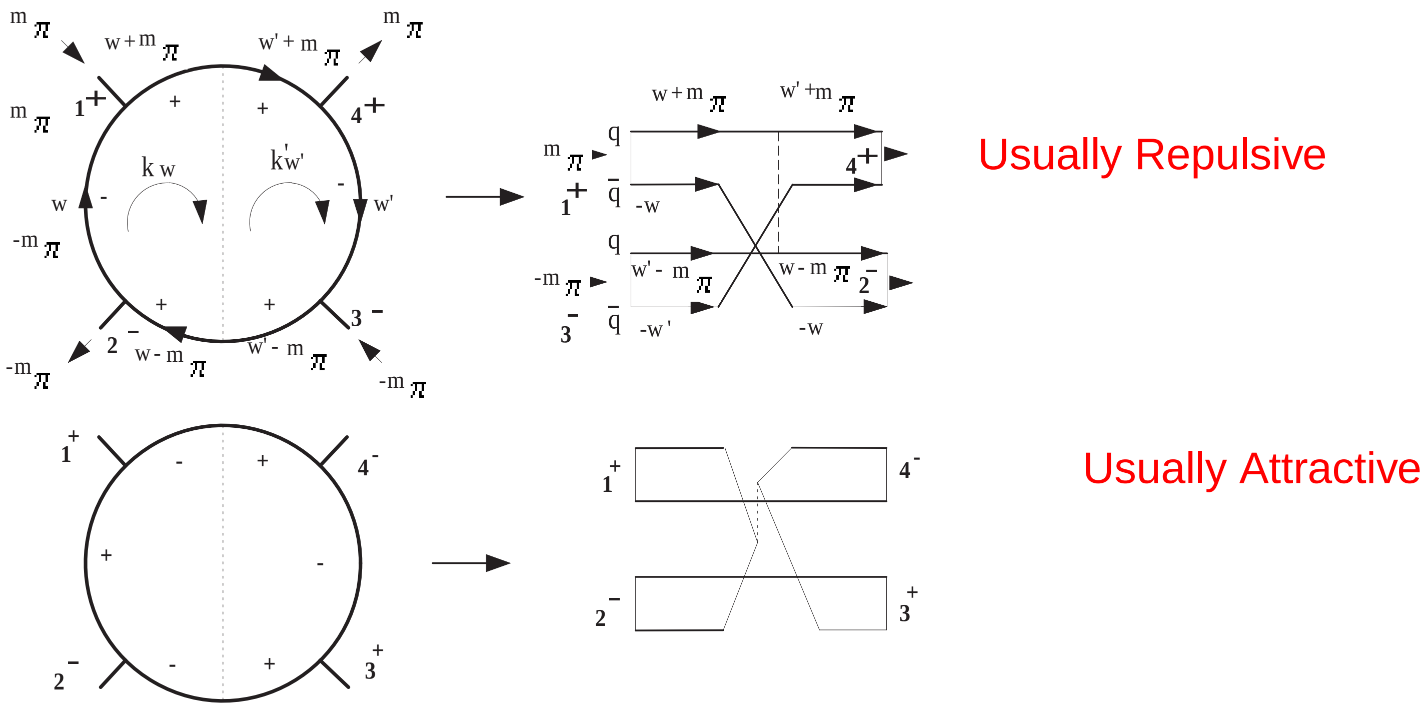

$$N[I]\begin{bmatrix}\Phi^{-2} & \Phi^{+2}\end{bmatrix}\begin{bmatrix}H^{++} & H^{+-}\\ H^{-+} & H^{--}\end{bmatrix}\begin{bmatrix}\Phi^{+2}\\ \Phi^{-2}\end{bmatrix}\rightarrow [N]\; m_\pi \begin{bmatrix}\Phi^{-2} & \Phi^{+2}\end{bmatrix}\sigma_3\begin{bmatrix}\Phi^{+}\\ \Phi^{-}\end{bmatrix}$$

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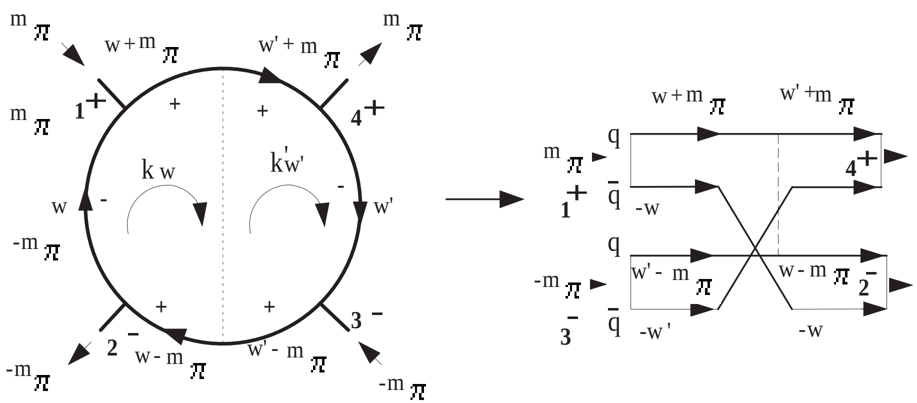
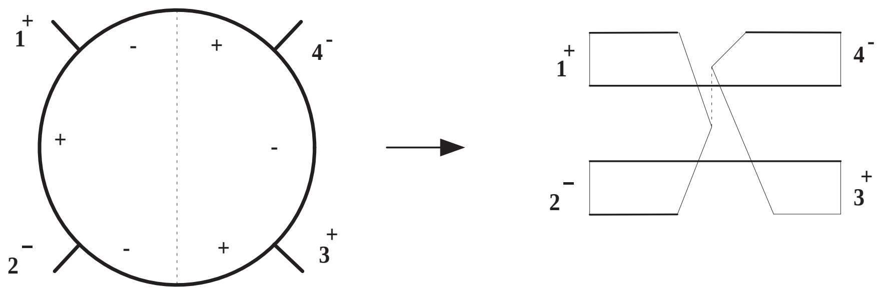


Usually Repulsive



Usually Attractive

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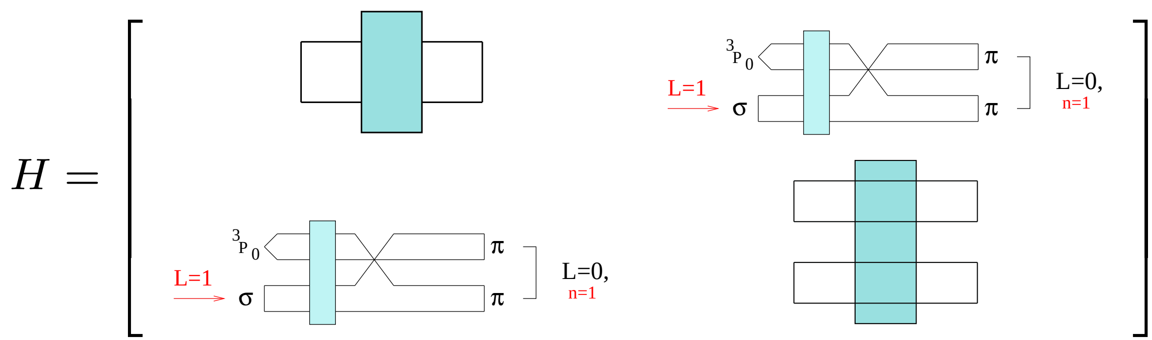
It turns out that it is possible to obtain the Weinberg results

$\sin\varphi \rightarrow 1, \ \left\{-7/2\,\frac{m_\pi^2}{f_\pi^2}, \quad 1\,\frac{m_\pi^2}{f_\pi^2}\right\}$ JER,Bicudo,Sczcepaniak,Cotanch,Maris

Low Energy scalar resonances

We want to solve a set of coupled channel equations



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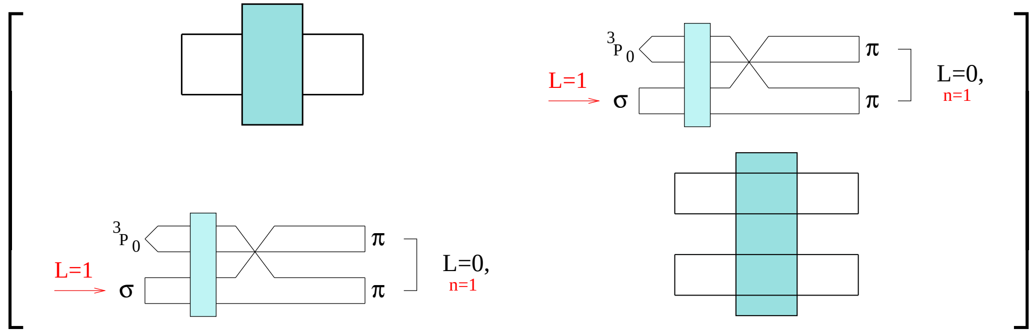
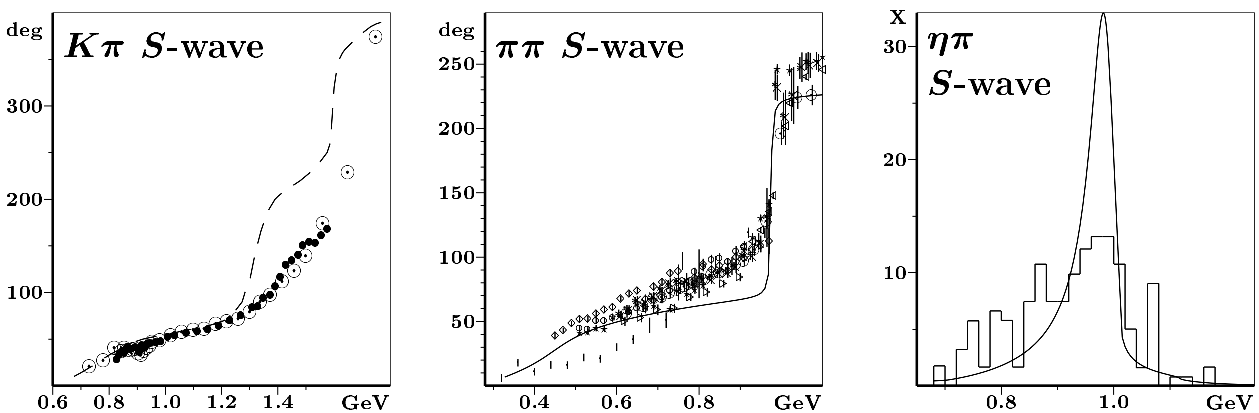
$H =$



- In 1986 we (with G. Rupp,E. Van Beveren et al)have obtained the following results,



$K\pi$ invariant mass — κ pole: $727 - i263$ MeV

$\pi\pi$ invariant mass — σ pole: $470 - i208$ MeV; f_0 pole: $994 - i20$ MeV

$\eta\pi$ invariant mass — a_0 pole: $962 - i28$ MeV

Entropy of the broken vacuum

- Consideration of finite temperatures amounts to evaluation of both the vacuum energy and the entropy of the system simultaneously and then to the minimization the free energy with respect to the order parameter: $S = -N_C N_f \sum_p \sum_{n=0}^{2} w_{np} \log w_{np}$

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- An example: the operator
$K = -\sum_p \left[\log w_{0p} + \frac{1}{2} \log\left(\frac{w_{1p}}{w_{0p}}\right) C_p^\dagger C_p + \frac{1}{2} \log\left(\frac{w_{2p}}{w_{1p}}\right) \left(C_p^\dagger\right)^2 C_p^2 \right]$ yields
$\langle 0[\varphi] | K | 0[\varphi] \rangle = S$

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- It is instructive to see how the entropy appears as a result of averaging over the vacuum of a local operator:$S = \langle 0[\varphi]|K|0[\varphi]\rangle$
- An example: the operator $K = -\sum_p \left[\log w_{0p} + \frac{1}{2} \log\left(\frac{w_{1p}}{w_{0p}}\right) C_p^\dagger C_p + \frac{1}{2} \log\left(\frac{w_{2p}}{w_{1p}}\right) \left(C_p^\dagger\right)^2 C_p^2 \right]$ yields $\langle 0[\varphi]|K|0[\varphi]\rangle = S$
- the thermal mass-gap equation guarantees balance between the energy and the entropy of the vacuum and can be derived by minimizing the free energy operator $H - TK$ averaged over the vacuum :$F_{\rm vac}[\varphi] = \langle 0[\varphi]|(H - TK)|0[\varphi]\rangle = E_{\rm vac} - TS$

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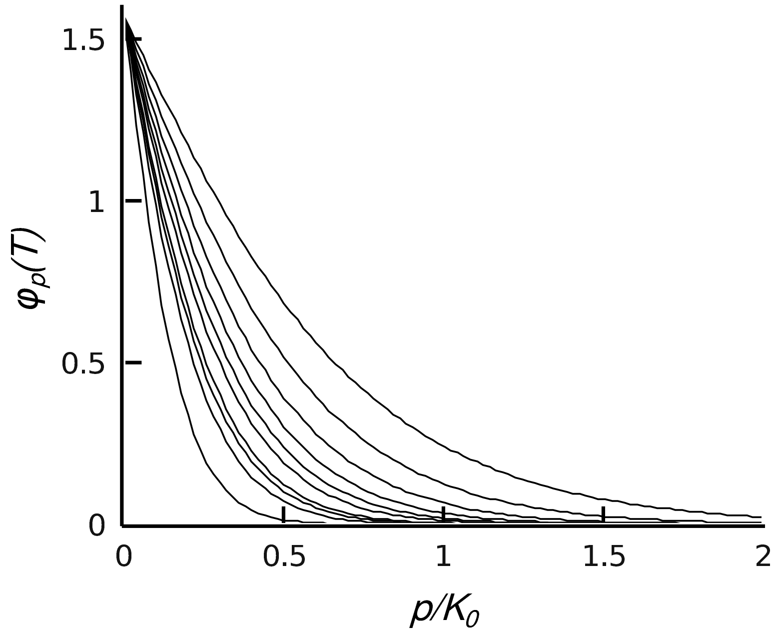
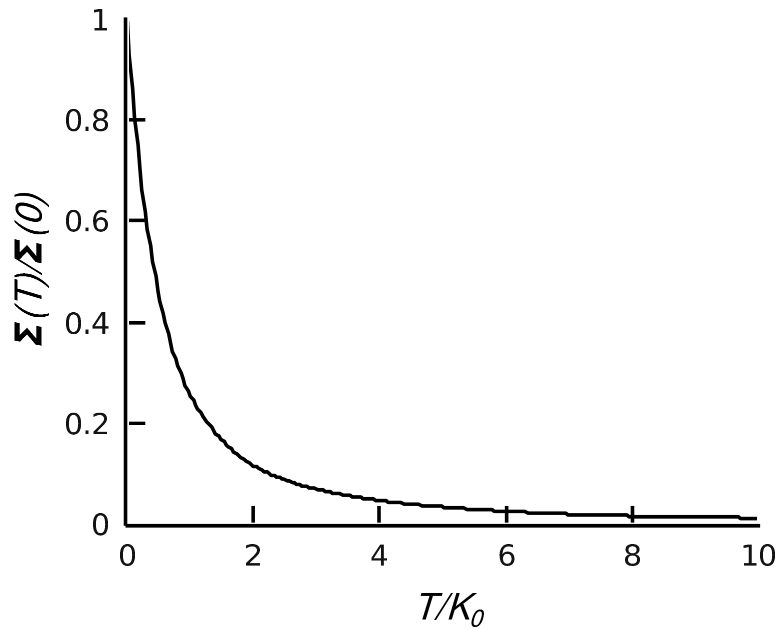
- Then the mass-gap equation is $\frac{\delta F_{\rm vac}[\varphi]}{\delta \varphi_p} = \frac{\delta E_{\rm vac}[\varphi]}{\delta \varphi_p} - T\frac{\delta S[\varphi]}{\delta \varphi_p} = 0$ which is nothing but a generalization of the zero-temperature mass-gap equation to finite temperatures.with A. Nefediev

Chiral angle dependence with T





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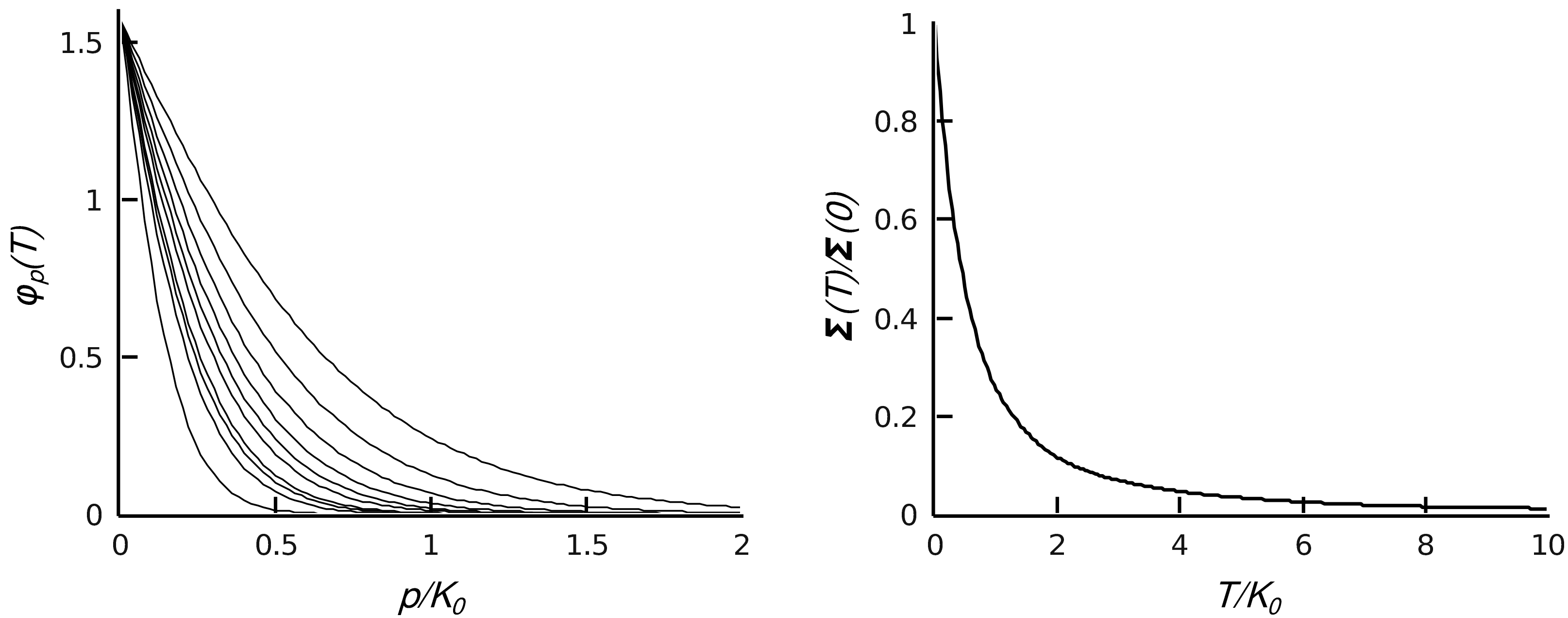

Summary

The role of Chiral symmetry in hadronic physics is,

- To ensure the existence of a Goldstone pion.
- To ensure that any microscopic calculation of $\pi - \pi$ scattering must get the Weinberg results...
- ... and more generally, to control pion mediated reactions.
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And also,

- Coupled channels are presumably to be blamed for the existence of light scalars: In this sector it is very difficult to escape from “pole doubling”
- Heating produces quark-antiquark pair evaporation and changes the Fock space